The Strategic Entering Time of a Commerce Platform

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Joint work with J. Kim and B. Kim

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Strategic Time of Arrivals to a Service System

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- We will extend the study to modern service systems, the online and e-commerce platforms.

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- Numbers of joining buyers and sellers are independent Poisson random variables with respective means λ_b and λ_s.

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- Balking and reneging are not allowed; every arrival will enter and stay until either being traded or time T.
- A buyer receives reward R_b from a trade and incurs waiting cost at rate C_b, and (R_s, C_s) for a seller.

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- Suppose the buyer chooses $B(t)$ as the distribution function of entering time, and $S(t)$ by the seller, that is, $B(t)$ is the strategy of buyer and $S(t)$ is the strategy of seller.
- **Proposition**. Under strategies B and S, with respective pdf's $b(t)$ and $s(t)$, numbers of joining buyers and sellers are independent nonhomogeneous Poisson processes with intensity functions $\lambda_b b(t)$ and $\lambda_s s(t)$, respectively.

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- Expected payoff of the seller is similarly defined.

Nash Equilibrium Strategy Profile

- (B, S) is a Nash equilibrium (NE) strategy profile if and only if

$$\pi_b(B|B,S) = \sup_{\widetilde{B}\in\mathcal{D}} \pi_b(\widetilde{B}|B,S),$$

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- In equilibrium, if a buyer enters at t and another enters at s, then expected payoffs must be identical at these two instants. Thus, for some τ_b and any $t \in (\tau_b, T)$,

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- Similarly, for some τ_s and any $t \in (\tau_s, T)$,

$$\frac{d}{dt}\pi_s(t|B,S) = 0.$$

Solutions of the Differential Equations

(i) For $t \in (\tau_b, T] \setminus \{0\}$,

$$B'(t) = \begin{cases} \dfrac{C_b}{\lambda_b} \dfrac{\mathbb{P}(X(\lambda_b B(t)) \geq Y(\lambda_s S(t)))}{R_b \mathbb{P}(X(\lambda_b B(t)) = Y(\lambda_s) - 1) + C_b \int_t^T \mathbb{P}(X(\lambda_b B(t)) = Y(\lambda_s S(u)) - 1) du}, \\ \text{if } t > 0, \\ \dfrac{C_b}{\lambda_b} \dfrac{1}{R_b \mathbb{P}(X(\lambda_b B(t)) = Y(\lambda_s) - 1) + C_b \int_0^T \mathbb{P}(X(\lambda_b B(t)) = Y(\lambda_s S(u)) - 1) du}, \\ \text{if } t < 0, \end{cases}$$

and $B(\tau_b) = 0$ and $B(T) = 1$.

(ii) For $t \in (\tau_s, T] \setminus \{0\}$,

$$S'(t) = \begin{cases} \dfrac{C_s}{\lambda_s} \dfrac{\mathbb{P}(X(\lambda_s S(t)) \geq Y(\lambda_b B(t)))}{R_s \mathbb{P}(X(\lambda_s S(t)) = Y(\lambda_b) - 1) + C_s \int_t^T \mathbb{P}(X(\lambda_s S(t)) = Y(\lambda_b B(u)) - 1) du}, \\ \text{if } t > 0, \\ \dfrac{C_s}{\lambda_s} \dfrac{1}{R_s \mathbb{P}(X(\lambda_s S(t)) = Y(\lambda_b) - 1) + C_s \int_0^T \mathbb{P}(X(\lambda_s S(t)) = Y(\lambda_b B(u)) - 1) du}, \\ \text{if } t < 0, \end{cases}$$

and $S(\tau_s) = 0$ and $S(T) = 1$.

Existence and Uniqueness of NE Strategy Profile

- The differential equations of $B'(t)$, $\tau_b < t < T$, and $S'(t)$, $\tau_s < t < T$, with $B(\tau_b) = S(\tau_s) = 0$ and $B(T) = S(T) = 1$ are the if-and-only-if condition of NE.

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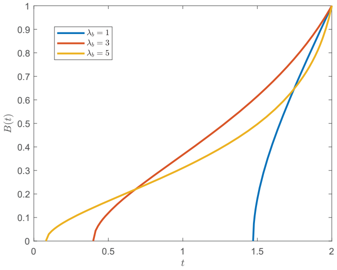
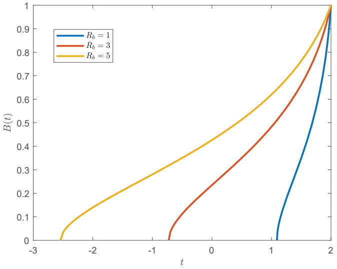
- **Theorem 1**. The NE strategy profile exists uniquely.
- Associated shapes of NE strategy (B, S) can shed insight on the best response of the buyers and sellers in the cooperative and non-cooperative game.

Illustration of Non-Cooperation Between Buyers

Supply is deterministic with size 2 and at time T:



(a) $R_b = 2$

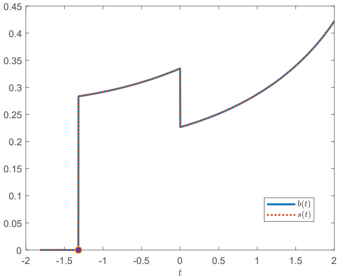


(b) $\lambda_b = 4$

With (a) a larger demand rate, competition becomes stronger, or
(b) a larger reward, it is more affordable,
the buyer tends to enter earlier for better chance to be traded, shows non-cooperation between buyers.

Illustration of Cooperation Between Buyers and Sellers

$R_b = R_s = 4$:



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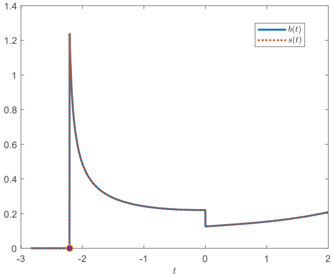
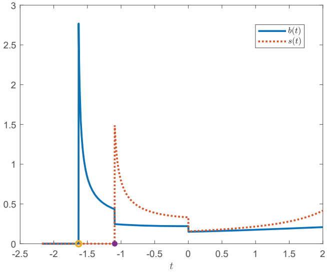


$\lambda_b = \lambda_s = 6$

1. When all parameters are the same, two strategies are identical shows cooperation between buyer and seller.
2. When both rates are larger, the competition results in a buyer (seller) tends to arrive earlier.

Illustration 1 of Mixed Non-Cooperate and Cooperate

$R_b = 4, R_s = 2$:



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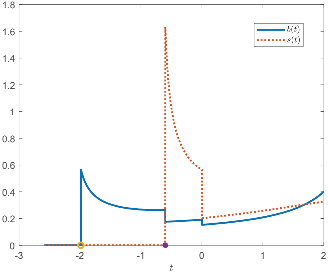
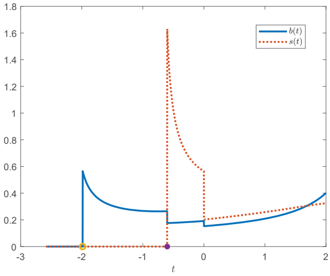


$\lambda_b = 6, \lambda_s = 4$

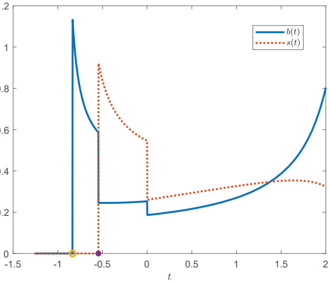
1. When the supply rate is smaller, an insecure buyer tends to arrive earlier while the secure seller enters later.
2. When $\max\{\tau_b, \tau_s\} < 0$, respective areas to the left of 0, i.e., probabilities of arriving before 0, are close.

Illustration 2 of Mixed Non-Cooperate and Cooperate

$\lambda_b = 6, \lambda_s = 4$:



$R_b = 4, R_s = 2$

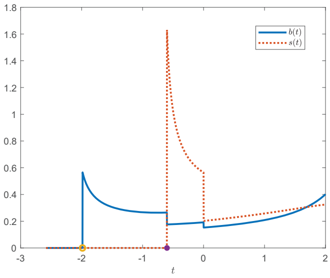


$R_b = 2, R_s = 2$

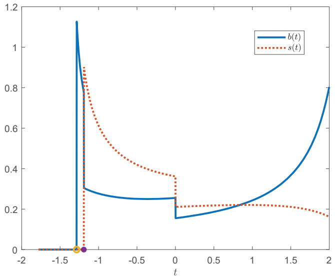
1. When the reward is reduced, a buyer can no longer afford to arrive early so that $b(t)$ is squeezed to the right.
2. While $s(t)$, even with λ_s and R_s unchanged, moves rightward a little accordingly.

Illustration 3 of Mixed Non-Cooperate and Cooperate

$\lambda_b = 6, \lambda_s = 4$:



(a) $R_b = 4, R_s = 2$



(b) $R_b = 2, R_s = 4$

When the demand rate is larger than the supply rate, comparing (a) and (b), i.e., reversing respective rewards, shows that a buyer is more concern on being traded than on waiting cost.

Social Welfare Under NE Strategy

- Let (B^*, S^*) be the unique NE. Associated social welfare is

$$W(B^*, S^*) = \lambda_b \pi_b(B^* | B^*, S^*) + \lambda_s \pi_s(S^* | B^*, S^*).$$

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- Since $\pi_b(B^* | B^*, S^*) = \pi_b(t | B^*, S^*)$ for all $t \in [\tau_b, T]$,

$$\pi_b(B^* | B^*, S^*) = \pi_b(T | B^*, S^*) = R_b \mathbb{P}\{X(\lambda_b) < Y(\lambda_s)\}.$$

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  where $X(\mu)$ and $Y(\nu)$ are independent Poisson random variables with means μ and ν, respectively.
- Similarly, $\lambda_s \pi_s(S^* | B^*, S^*) = R_s \mathbb{P}\{X(\lambda_b) > Y(\lambda_s)\}$, and, thus,

$$W(B^*, S^*) = \lambda_b R_b \mathbb{P}\{X(\lambda_b) < Y(\lambda_s)\} + \lambda_s R_s \mathbb{P}\{X(\lambda_b) > Y(\lambda_s)\}.$$

Social Welfare Under Socially Optimal Strategy

- Let (B^o, S^o) be the socially optimal strategy profile under centralization. It can be easily seen

$$B^o(t) = S^o(t) = \begin{cases} 0 & \text{if } t < T, \\ 1 & \text{if } t = T, \end{cases}$$

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  that is, all buyers and sellers enter on T to avoid waiting.
- Thus, the social welfare under (B^o, S^o) is

$$W(B^o, S^o) = (R_b + R_s)\mathbb{E}[\min\{X(\lambda_b), Y(\lambda_s)\}].$$

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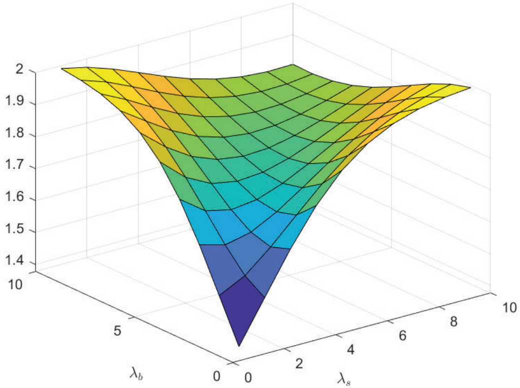
- As a measure for inefficiency of decentralization, PoA ≥ 1.
- For our model,

$$\begin{aligned}\mathrm{PoA} &= \frac{W(B^o, S^o)}{W(B^*, S^*)} \\ &= \frac{(R_b + R_s)\mathbb{E}[\min\{X(\lambda_b), Y(\lambda_s)\}]}{\lambda_b R_b \mathbb{P}(X(\lambda_b) < Y(\lambda_s)) + \lambda_s R_s \mathbb{P}(X(\lambda_b) > Y(\lambda_s))}.\end{aligned}$$

Note that PoA is independent of C_b, C_s and T.

Illustrations of PoA

$R_b = 3$ and $R_s = 3$, with λ_b and λ_s varying:



The infimum of PoA occurs at both λ_b and λ_s approaching 0.

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- That is, PoA $= 1$!

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- Under RD, the trade is matched by a buyer and a seller from random draws from respective queues.
- As RD would also decrease the motivation of joining early, yet not as much as by LJFT, we have by Theorem 2 that under RD,

$$1 = PoA_{LJFT} < PoA_{RD} < PoA_{FJFT}.$$

Final Remarks

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- Experiments of this type have been conducted – Rapoport *et. al*. [2004], where aggregate behavior of participants was remarkably similar to the theoretical equilibrium prediction.
- Rational learning is closely related to development of game theory, convex optimization, and machine learning. It is an essential complement of analytic results that can make them practical and realized.

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- Rational learning is closely related to development of game theory, convex optimization, and machine learning. It is an essential complement of analytic results that can make them practical and realized.
- It would be interesting and useful to construct a learning strategy that would induce players of the game to make decisions gradually toward the NE strategy.

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Thanks for Your Attention